SOLUTION OF THERMAL CONDUCTIVITY PROBLEMS WITH BOUNDARY CONDITION OF THE THIRD KIND AND ARBITRARY COORDINATE AND TIME DEPENDENCE OF THE BIOT NUMBER

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UDC 536.24

An analytical solution of the thermal conductivity problem with boundary conditions of the third kind and arbitrary coordinate and time dependence of the Biot number is found in the form of a converging series of quadratures.

At the present time, much attention is being given to methods for solution of thermal conductivity problems with boundary conditions of the third kind and heat-liberation coefficient (Biot number) which varies over the surface and with time. A number of studies [1-6] have been dedicated to this question. However, those authors considered either stationary regimes with heat liberation dependent only on coordinates [1-3], or nonstationary regimes with heat-liberation coefficient dependent solely on time [4-6].

In practice, problems are often found where in calculating transitional temperature fields one must consider not only nonuniformity of the heat-liberation coefficient over the surface, but also time dependence of this quantity. Among such problems are calculation of nonstationary temperature fields with a local heat-liberation crisis, pulsations in the heat-liberation coefficient, accidental disruption of the heat-liberation process in the active zone of a reactor, etc.

The method to be presented in this study permits solution of the thermal conductivity problem with arbitrary space—time dependence of the heat-liberation coefficient (Biot number). The principle of the method, consisting of representing the desired solution in the form of a superposition of solutions for constant Biot number, was used previously in [1] to solve a stationary thermal conductivity problem.

We will consider an infinite heat-liberating plate, whose nonstationary dimensionless temperature field is described by the thermal conductivity equation in cylindrical coordinates

$$\frac{\partial\Theta}{\partial \mathrm{Fo}}-\frac{1}{\rho}\frac{\partial\Theta}{\partial\rho}-\frac{\partial^2\Theta}{\partial\rho^2}-\frac{\partial^2\Theta}{\partial z^2}=Q(\rho,\ z,\ \mathrm{Fo}), \tag{1}$$

with initial condition

$$\Theta(\rho,\ z,\ \mathrm{Fo})|_{\mathrm{Fo}=0}=\Theta_0(\rho,\ z), \tag{2}$$

symmetry condition

$$\left.\frac{\partial\Theta}{\partial\rho}\right|_{\rho=0}=0 \tag{3}$$

and boundary conditions of the third kind on the surfaces z = 0 and z = 1 for Fo > 0:

$$\left(\frac{\partial\Theta}{\partial z}-\mathrm{Bi}_1(\rho,\ \mathrm{Fo})\,\Theta\right)_{z=0}=0, \tag{4}$$

$$\left(\frac{\partial\Theta}{\partial z}+\mathrm{Bi}_{02}\Theta\right)_{z=1}=0, \tag{5}$$

where

$$\mathrm{Bi}_1(\rho,\ \mathrm{Fo})=\begin{cases}\mathrm{Bi}_c(\rho,\ \mathrm{Fo}), & \rho\leqslant a;\\ \mathrm{Bi}_{01}, & \rho>a;\end{cases} \tag{6}$$

Institute of Theoretical and Experimental Physics, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 41, No. 3, pp. 536-540, September, 1981. Original article submitted June 25, 1980.

 0022-0841/81/4103-1032$07.50 

$Bi_{01}$, $Bi_{02}$ are constants; $Bi_c(\rho, Fo)$ is an arbitrary function of $\rho$ and Fo. We represent the unknown $\Theta(\rho, z, Fo)$ in the form

$$\Theta(\rho, z, \mathrm{Fo}) = \Theta_0(\rho, z, \mathrm{Fo}) + \overline{\Theta}(\rho, z, \mathrm{Fo}), \tag{7}$$

where $\Theta_0(\rho, z, Fo)$ is a solution of the nonstationary boundary problem Eqs. (1)-(6) for constant Biot numbers:

$$\mathrm{Bi}_1(\rho, \mathrm{Fo}) \equiv \mathrm{Bi}_{01} = \mathrm{const}_1, \quad \mathrm{Bi}_{02} = \mathrm{const}_2,$$

while $\overline{\Theta}(\rho, z, Fo)$ is a "perturbation" of the temperature field $\Theta_0(\rho, z, Fo)$ produced by the variation of the Biot number, and described by the following boundary problem:

$$\frac{\partial\overline{\Theta}}{\partial \mathrm{Fo}} - \Delta\overline{\Theta} = 0, \tag{8}$$

$$\overline{\Theta}\big|_{\mathrm{Fo}=0} = 0, \tag{9}$$

$$\left.\frac{\partial\overline{\Theta}}{\partial\rho}\right|_{\rho=0} = 0, \tag{10}$$

$$\left(\frac{\partial\overline{\Theta}}{\partial z} - \mathrm{Bi}_{01}\overline{\Theta}\right)_{z=0} = (\mathrm{Bi}_1(\rho, \mathrm{Fo}) - \mathrm{Bi}_{01})(\Theta_0 + \overline{\Theta})_{z=0}, \tag{11}$$

$$\left(\frac{\partial\overline{\Theta}}{\partial z} + \mathrm{Bi}_{02}\overline{\Theta}\right)_{z=1} = 0. \tag{12}$$

Determination of the function $\Theta_0(\rho, z, Fo)$ is not difficult, so we will assume it to be known, and seek a solution of Eqs. (8)-(12) in the form

$$\overline{\Theta}(\rho, z, \mathrm{Fo}) = \sum_{i=1}^{\infty}\overline{\Theta}_i(\rho, z, \mathrm{Fo}), \tag{13}$$

where $\overline{\Theta}_i(\rho, z, Fo)$ satisfies the boundary problem Eqs. (8)-(12) with boundary conditions on the surface z = 0 of the form

$$\left(\frac{\partial\overline{\Theta}_i}{\partial z} - \mathrm{Bi}_{01}\overline{\Theta}_i\right)_{z=0} = (\mathrm{Bi}_1(\rho, \mathrm{Fo}) - \mathrm{Bi}_{01})\left.\overline{\Theta}_{i-1}\right|_{z=0} \ (\text{for } i = 1\ \overline{\Theta}_0 \equiv \Theta_0). \tag{14}$$

It is obvious that in the case of convergence (i.e., when the limit $\lim_{i\to\infty}\overline{\Theta}_i = 0$ is satisfied) series (13) also satisfies condition (11), i.e., is the desired solution. We will return to the question of convergences later, assuming meanwhile that series (13) does converge. We write $\overline{\Theta}_i(\rho, z, Fo)$ in the form

$$\overline{\Theta}_i(\rho, z, \mathrm{Fo}) = \sum_{j=1}^{3} U_{i,j}(\rho, z, \mathrm{Fo}),$$

where $U_{i,1}$ is a solution of the Laplace equation $\Delta U_{i,1} = 0$, satisfying the inhomogeneous boundary condition

$$\left(\frac{\partial U_{i,1}}{\partial z} - \mathrm{Bi}_{01}U_{i,1}\right)_{z=0} = (\mathrm{Bi}_1(\rho, \mathrm{Fo}) - \mathrm{Bi}_{01})\left.\overline{\Theta}_{i-1}\right|_{z=0};$$

$U_{i,2}$ is the solution of the homogeneous thermal conductivity equation

$$\frac{\partial U_{i,2}}{\partial \mathrm{Fo}} - \Delta U_{i,2} = 0$$

for nonzero initial condition $U_{i,2}|_{Fo=0} = -U_{i,1}|_{Fo=0}$; $U_{i,3}$ is the solution of the inhomogeneous thermal conductivity equation

$$\frac{\partial U_{i,3}}{\partial \mathrm{Fo}} - \Delta U_{i,3} = -\frac{\partial U_{i,1}}{\partial \mathrm{Fo}}$$

with zero initial condition $U_{i,3}|_{Fo=0} = 0$.

Moreover, the functions $U_{i,j}$ must satisfy the conditions

$$\left(\frac{\partial U_{i,j}}{\partial z}+\mathrm{Bi}_{02}U_{i,j}\right)_{z=1}=0,\qquad \left.\frac{\partial U_{i,j}}{\partial \rho}\right|_{\rho=0}=0.$$

Omitting intermediate expressions for $U_{i,j}$, which present no difficulty in determination (e.g., by the method of separation of variables), we finally write

$$\overline{\Theta}_i(\rho,\ z,\ \mathrm{Fo})=\int_0^{\mathrm{Fo}}\int_0^{a}(\mathrm{Bi}_{01}-\mathrm{Bi}_1(\rho',\ \mathrm{Fo}'))\,\overline{\Theta}_{i-1}(\rho',\ z,\ \mathrm{Fo}')\big|_{z=0}\,G(\rho,\ \rho',\ z,\ \mathrm{Fo},\ \mathrm{Fo}')\,d\rho' d\,\mathrm{Fo}', \tag{15}$$

where

$$G(\rho,\ \rho',\ z,\ \mathrm{Fo},\ \mathrm{Fo}')=\int_0^{\infty}J_0(\lambda\rho)\,J_0(\lambda\rho')\,\lambda\rho'\sum_{n=1}^{\infty}P_{\lambda,\mu_n}Z_n(z)\exp\left((\lambda^2+\mu_n^2)(\mathrm{Fo}'-\mathrm{Fo})\right)d\lambda;$$

$$P_{\lambda,\mu_n}=\frac{2\mu_n^2}{\mathrm{Bi}_{01}+\mu_n^2+\mathrm{Bi}_{01}^2+\mathrm{Bi}_{02}S_n}\left\{1+\left\{2\lambda\exp\lambda\left[\left(\mu_n-\frac{\mathrm{Bi}_{01}\mathrm{Bi}_{02}}{\mu_n}\right)\sin\mu_n-(\mathrm{Bi}_{01}+\mathrm{Bi}_{02})\cos\mu_n\right]\right\}\times\right.$$

$$\left.\times[(\mathrm{Bi}_{01}+\mathrm{Bi}_{02})(1+\exp(2\lambda))\,\lambda+(\lambda^2+\mathrm{Bi}_{01}\mathrm{Bi}_{02})(\exp(2\lambda)-1)]^{-1}\right\};$$

$$Z_n(z)=\cos(\mu_n z)+\frac{\mathrm{Bi}_{01}}{\mu_n}\sin(\mu_n z);\qquad S_n=\frac{\mu_n^2+\mathrm{Bi}_{01}^2}{\mu_n^2+\mathrm{Bi}_{02}^2};$$

and $\mu_n$ are the roots of the transcendental equation

$$\mathrm{tg}\,\mu_n=\frac{\mu_n(\mathrm{Bi}_{01}+\mathrm{Bi}_{02})}{\mu_n^2-\mathrm{Bi}_{01}\mathrm{Bi}_{02}}.$$

Thus, the solution of Eqs. (8)-(12) for the temperature perturbation produced by variation of the Biot number can be expressed in the form of series (13), the terms of which are defined by Eq. (15). It is clear that the convergence of this series depends on $\mathrm{Bi}_{01}$, $\mathrm{Bi}_{02}$, $a$, and the form of the function $\mathrm{Bi}_1(\rho,\ \mathrm{Fo})$, and must be considered individually in each concrete case.

However, it is simple to prove that upon fulfillment of the condition $\mathrm{Bi}_{01}\geqslant\mathrm{Bi}_1(\rho,\ \mathrm{Fo})$ series (13) converges unconditionally. In fact, $\overline{\Theta}_i(\rho,\ z,\ \mathrm{Fo})$ described by Eqs. (8)-(12) with boundary condition (14) has the sense of the temperature produced by surface heat sources with density $(\mathrm{Bi}_{01}-\mathrm{Bi}_1(\rho,\ \mathrm{Fo}))\overline{\Theta}_{i-1}(\rho,\ z,\ \mathrm{Fo})|_{z=0}$. It is obvious that with increasing source density, duration of action, and increase in surface area encompassed by these sources, the temperature will also rise.

Thus, we may write

$$\overline{\Theta}_i(\rho,\ z,\ \mathrm{Fo})<\varphi_i(z)\leqslant\varphi_i^{\max}\equiv\varphi_i(z=0), \tag{16}$$

where $\varphi_i(z)$ is nothing but the one-dimensional stationary temperature field produced by action of surface sources with z = 0 and density $(\mathrm{Bi}_{01}-\mathrm{Bi}_1^{\min})\varphi_{i-1}^{\max}$ ($i = 1, 2, 3, \ldots$); $\mathrm{Bi}_1^{\min}$ is the minimum value of the function $\mathrm{Bi}_1(\rho,\ \mathrm{Fo})$ $(0\leqslant\rho<a,\ 0\leqslant\mathrm{Fo}<\infty)$; $\varphi_0^{\max}\equiv\Theta_0^{\max}$ is the maximum value of the function $\Theta_0(\rho,\ z,\ \mathrm{Fo})|_{z=0}$ $(0\leqslant\rho\leqslant a,\ 0\leqslant\mathrm{Fo}<\infty)$.

From solution of the corresponding problem for $\varphi_i(z)$ one easily finds that

$$\frac{\varphi_i^{\max}}{\varphi_{i-1}^{\max}}=\frac{(\mathrm{Bi}_{01}-\mathrm{Bi}_1^{\min})(1+\mathrm{Bi}_{02})}{\mathrm{Bi}_{01}\left(1+\mathrm{Bi}_{02}+\dfrac{\mathrm{Bi}_{02}}{\mathrm{Bi}_{01}}\right)}<1.$$

Then, according to D'Alembert's principle the positive-sign numerical series $\sum_{i=1}^{\infty}\varphi_i^{\max}$ converges so that the functional series $\sum_{i=1}^{\infty}\Theta_i(\rho,\ z,\ \mathrm{Fo})$ for which the numerical series $\sum_{i=1}^{\infty}\varphi_i^{\max}$ is the major also converges in view of Eq. (16). We note that series (13) can always be made unconditionally convergent by using for $\mathrm{Bi}_{01}$ in the solution for $\Theta_0(\rho,\ z,\ \mathrm{Fo})$ the maximum value of the function $\mathrm{Bi}_1(\rho,\ \mathrm{Fo})$ $(0\leqslant\rho<\infty,\ \mathrm{Fo}<\infty)$.

It should be noted that the approach used in solving this problem can be applied to determination of nonstationary temperature fields produced by variation of the Biot number over coordinate and time in bodies of any form.

## NOTATION

$\rho$, z, dimensionless coordinates; $\Theta$, dimensionless temperature; Q, dimensionless volume heat-liberation density per unit time; $Fo = \varkappa\tau/\delta^2$, Fourier number; $Bi_1(\rho, Fo) = \alpha(\rho, Fo)\cdot\delta/\lambda$, Biot number; $\varkappa$, thermal diffusivity coefficient; $\delta$, plate thickness; $\tau$, time; $\alpha(\rho, Fo)$, heat-liberation coefficient; $\lambda$, thermal conductivity coefficient; i, summation index; $J_0$, zero order Bessel function of the first kind.

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# HEAT TRANSFER OF A SEMITRANSPARENT PLATE UNDER CONDITIONS OF A REGULAR REGIME OF THE SECOND KIND


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UDC 536.33:536.241



The kinetics of the cooling of a plane semitransparent layer under conditions of a regular regime of the second kind is analyzed.


Heating and cooling of optical blanks made of glass and ceramics quite often are carried out under conditions of linearly varying ambient temperature. In this case the calculation of the rate of cooling and the temperature drops in the glass, in practice, is carried out according to empirical relations [1, 2] or according to dependencies of a regular regime of the second kind [3]. Moreover, it is known that at sufficiently high temperatures in glass and diathermic ceramics, bulk self-radiation of the material develops, and use of the ordinary Fourier equation is not very effective here. The features of the heating of a semitransparent layer under conditions of a regular regime of the first kind were investigated in [4].

Study [5] is the only work in which an analysis is given of the features of a regular regime of the second kind in semitransparent materials on the basis of the exact equations of radiant—conductive heat exchange. However, we should note that an analysis was performed for a narrow range of parameters of the regime and at low temperatures, when the radiant component of heat exchange appeared weak. Therefore, the conclusion of the authors concerning the existence of a regular regime of the second kind in semitransparent materials, analogous to the situation in the classical theory of heat conduction, has little substantiation.

The present study is devoted to a theoretical study of the features of the unsteady process of radiation—conduction heat exchange under conditions of a regular regime of the second kind, when the radiation component makes a noticeable contribution to the value of the total heat flux.

We consider the problem in the following formulation. A plate of glass of thickness 2d with optically smooth surfaces, the transfer of heat within which occurs simultaneously by


Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 41, No. 3, pp. 541-546, September, 1981. Original article submitted June 16, 1980.

0022-0841/81/4103-1035$07.50